SCHRÖDINGER EQUATION IN QFD REPRESENTATION WITH NAVIER - STOKES DISSIPATION

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ABSTRACT

1. Introduction. The Quantum Fluid Dynamics (QFD) representation of quantum mechanics was partly motivated by questions of the completeness of the quantum theory, while admitting to its internal consistency. The QFD representation has its foundations in Madelung's work (1926) in the early times and followed by Bohm's interpretation of quantum mechanics (1950's) with the goal to find classically identifiable dynamical variables at the sub-particle level [1]. The approach leads to conservation laws for "mass", "momentum" and "energy" similar to those in hydrodynamics for a compressible, non dissipative fluid.

The QFD equations are a set of nonlinear partial differential equations. In this sense, they may be seen as a step in the negative direction as compared to the linear Schrodinger equation. However, in this scheme, the oscillatory real and imaginary components of the complex wave function are replaced by the monotonous the amplitude and phase. This has been exploited as a significant advantage in computational quantum mechanics [1, 2].

In this study, the nonlinear field equations are suggested as the natural framework for studying fundamentally nonlinear phenomena such as solitons and chaos, in a multi-dimensional context. This exposé extends the QFD formalism to the case of a non-isolated medium. The interaction of this analogous fluid medium with the environment is postulated to lead to a dissipative piece in the acceleration and is introduced phenomenologically as in the classical Navier - Stokes equation.

The Schrödinger equation being linear and having basic conservation laws for momentum and energy does not show possibilities for "turbulence". As for the Navier - Stokes equation, this formulation "**may**" contain a necessary ingredient of chaotic behavior for quantum mechanics. This modified form allows for vorticities that may break into a cascade, eventually leading to "turbulence", as in the classical Navier - Stokes equation.

2. From Schrödinger's to Quantum Fluid Dynamics (QFD) Equation. The starting point is the Schrödinger equation for the complex valued wave function ψ, nondimensionalized by reducing the Planck's constant $\hbar$ of the conventional quantum mechanics and the mass m of the particle to unity is:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2}\nabla^2\psi + V\psi$$

$V = V(x,t)$ is the externally applied potential energy that guides the dynamics and $i = \sqrt{-1}$ is the usual imaginary unit.

The complex wave function ψ is represented by its amplitude A and phase S as:

$$\psi(x,t) = A(\mathbf{x},t)\ e^{i[S(\mathbf{x},t)-Et]}$$

Above by reference to quantum mechanics, E is singled out to be identified as at the total energy in the derivations below.

The substitution of the representation of the wave function in the polar form in the Schrodinger equation, the separation of its respectively imaginary and real parts and the definition of the density ρ, velocity $\mathbf{v}$, the externally applied force $\mathbf{g} = -\nabla V$ and the "Quantum potential" or "pressure" yield the hydrodynamics representation of quantum mechanics. The equations are respectively "Mass conservation" and "Momentum conservation" equations:

$$\rho = A^2 \qquad \mathbf{v} = \frac{\nabla S}{m} \qquad V_q = -\frac{1}{2}\frac{\nabla^2 A}{A} \qquad \mathbf{g} = -\nabla V$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} = -\nabla V_q + \mathbf{g}$$

These are called alternatively as "Madelung's quantum hydrodynamics equations", "Bohmian representation of quantum mechanics", "Quantum fluid dynamics (QFD) equations"...

The energy equation is also derived as an alternative to the momentum equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + V + V_q = E$$

S is interpreted as an action whose time rate S_t is an energy; $\frac{1}{2}m\mathbf{v} \cdot \mathbf{v} + V$ form the classical energy with its kinetic and potential energy components; and V_q as a non-local "quantum potential" as the Laplacian connects neighboring regions.

The linear and nonlinear parts of the momentum equation are the local and convective acceleration terms of classical fluid continua as in the Navier - Stokes equations. The acceleration in the Navier - Stokes equation contains also a duffisive component represented by the phenomenological term $-\mu\nabla^2\mathbf{v}$. μ is callesd as "viscosity". Adding this term to the momentum equation above leads to:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} = -\nabla(V + V_q) - \mu\nabla^2\mathbf{v}$$

3. Further use of QFD. The same decomposition of the wave funtion can also be applied to the Nonlinear Schrödinger equation that leads to the same mass conservation equation and slightly modified momentum and energy equations:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\nabla^2\psi - k^2|\psi|^2\psi) \quad \rightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} = -\nabla(V_q + k^2\rho)$$

$$\frac{\partial S}{\partial t} + (\frac{1}{2}|\mathbf{v}|^2 + V_p + k^2\rho) = E$$

QFD allows also the representation Gross - Pitaevskii equation for Bose-Einstein condensation.

References

[1] (Excellent review of the field) Robert E. Wyatt, Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics, Springer-Verlag NY, 2005.

[2] (Sample of the Work by AA's contributions over the years)
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